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Rings whose free modules satisfy the ascending chain condition on submodules with a bounded number of generators

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Abstract

Let R be a ring such that every finitely generated free (respectively, every free) right R-module satisfies the ascending chain condition on n-generated submodules for every positive integer n; then any ring Morita equivalent to R has the same property. This is in contrast to rings R which satisfy the ascending chain condition on n-generated right ideals, for some fixed positive integer n, for in this case rings Morita equivalent to R need not have the same property. If R is a right and left Ore domain and n is a positive integer such that the free right R-module $R_R^{(n)}$ satisfies the ascending chain condition on n-generated submodules then so too does every free right R-module. Many examples are given of rings for which every finitely generated free (respectively, every free) right module satisfies the ascending chain condition on n-generated submodules, for some positive integer n. © 1998 Elsevier Science B.V.

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1. Morita equivalence

Throughout this note, all rings are associative with identity and all modules are unital right modules. Let n be a positive integer. We say that a module M *satisfies n-acc* if every ascending chain of n-generated submodules terminates. If the module M satisfies n-*acc* for every positive integer n, then we shall say that M *satisfies pan-acc*. We shall say that the ring R *satisfies right n-acc* (respectively, *right pan-acc*) if the right

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R-module R satisfies *n-acc* (*pan-acc*). For information about any terms used without explanation, see [1] or [9].

Let R be any ring and let m, n be positive integers. Let $\mathscr{M}_n(R)$ denote the ring of all $n \times n$ matrices with entries in R and let $\mathscr{M}_{m\times n}(R)$ denote the additive Abelian group of all $m \times n$ matrices with entries in R. Let S denote the ring $\mathscr{M}_n(R)$. Clearly $\mathscr{M}_{m\times n}(R)$ is a right S-module with respect to matrix multiplication. Given elements $a_{ij} \in R$ $(1 \le i \le m,\ 1 \le j \le n)$, let (a_{ij}) denote the $m \times n$ matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

in $\mathscr{M}_{m\times n}(R)$.

Let F denote the free right R-module $R_R^{(m)}$. Let N and L be any n-generated R-submodules of F. There exist $a_{ij}, b_{ij} \in R$ $(1 \le i \le m,\ 1 \le j \le n)$ such that

$$N = (a_{11}, \dots, a_{m1})R + \cdots + (a_{1n}, \dots, a_{mn})R,$$

and

$$L = (b_{11}, \dots, b_{m1})R + \cdots + (b_{1n}, \dots, b_{mn})R.$$

Lemma 1.1. *With the above notation, $N \subseteq L$ if and only if there exists (c_{ij}) in S such that $(a_{ij}) = (b_{ij})(c_{ij})$.*

Proof. $N \subseteq L$ if and only if there exist elements $c_{ij} \in R$ $(1 \le i,\ j \le n)$ such that

$$(a_{1j}, \dots, a_{mj}) = (b_{11}, \dots, b_{m1})c_{1j} + \cdots + (b_{1n}, \dots, b_{mn})c_{nj},$$

for each $1 \le j \le n$, and this holds if and only if

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}. \quad \square$$

With N as above, we define $\alpha(N) = (a_{ij})S$, i.e. $\alpha(N)$ is the set of $m \times n$ matrices over R such that the transpose of each column is in N. Note that by Lemma 1.1, $\alpha(N)$ is independent of the choice of n-generating set for N. Moreover, Lemma 1.1 gives at once:

Corollary 1.2. *With the above notation, let $N \subseteq L$ be n-generated R-submodules of F. Then $\alpha(N) \subseteq \alpha(L)$.*

Now suppose that $a = (a_{ij}) \in \mathscr{M}_{m\times n}(R)$, where $a_{ij} \in R$ $(1 \le i \le m,\ 1 \le j \le n)$. We define

$$\beta(aS) = (a_{11}, \dots, a_{m1})R + \cdots + (a_{1n}, \dots, a_{mn})R.$$

Note that Lemma 1.1 shows that if $a, b \in \mathcal{M}_{m\times n}(R)$ with $aS = bS$ then $\beta(aS) = \beta(bS)$. Thus, for each a in $\mathcal{M}_{m\times n}(R)$, $\beta(aS)$ is a well-defined n-generated R-submodule of F. Moreover, Lemma 1.1 gives at once:

Corollary 1.3. *Let $a, b \in \mathcal{M}_{m\times n}(R)$ with $aS \subseteq bS$. Then $\beta(aS) \subseteq \beta(bS)$.*

Lemma 1.4. *With the above notation, $\beta\alpha(N) = N$ for every n-generated R-submodule N of F and $\alpha\beta(aS) = aS$ for every $a \in \mathcal{M}_{m\times n}(R)$.*

Proof. Clear. □

Theorem 1.5. *Let R be any ring and let m and n be positive integers. Then the free right R-module $R_R^{(m)}$ satisfies n-acc if and only if the right $\mathcal{M}_n(R)$-module $\mathcal{M}_{m\times n}(R)$ satisfies 1-acc.*

Proof. Clear by the above results. □

Corollary 1.6. *Let R be any ring and let n be any positive integer. Then the ring $\mathcal{M}_n(R)$ satisfies right 1-acc if and only if the free right R-module $R_R^{(n)}$ satisfies n-acc.*

Proof. Take $m = n$ in the theorem. □

Corollary 1.7. *Let R be any ring and let n be any positive integer such that the ring $\mathcal{M}_n(R)$ satisfies right 1-acc. Then R satisfies right n-acc.*

Proof. By Corollary 1.6. □

Heinzer and Lantz [7, Section 4] show that for every positive integer n there exists a commutative ring R_n such that R_n satisfies *n-acc* but R_n does not satisfy $(n+1)$-*acc*. Thus $\mathcal{M}_{n+1}(R_n)$ does not satisfy 1-*acc* (Corollary 1.7). This shows that for any positive integer n, matrix rings over rings which satisfy right *n-acc* need not themselves satisfy right *n-acc*, and in particular "satisfying right *n-acc*" is not a Morita invariant.

Let R be any ring and let $S = \mathcal{M}_n(R)$, for any positive integer n. For each $1 \le i, j \le n$, let e_{ij} denote the matrix in S with (i,j)th entry 1 and all other entries 0. Let F be a free right S-module with basis $\{f_\lambda : \lambda \in \Lambda\}$. Then F is a free right R-module with basis $\{f_\lambda e_{ij} : \lambda \in \Lambda,\ 1 \le i, j \le n\}$, and if N is any m-generated S-submodule of F, say $N = x_1 S + \cdots + x_m S$ then N is an mn^2-generated R-submodule of F, because $N = \sum_i \sum_j \sum_k x_k e_{ij} R$. This gives the following result.

Lemma 1.8. *Let R be any ring such that every (finitely generated) free right R-module satisfies pan-acc. Let n be any positive integer. Then every (finitely generated) free right $\mathcal{M}_n(R)$-module satisfies pan-acc.*

Theorem 1.9. *The following statements are equivalent for a ring R:*

(i) *For each positive integer n, the ring $\mathcal{M}_n(R)$ satisfies right pan-acc.*

(ii) *For each positive integer n, the ring $\mathcal{M}_n(R)$ satisfies right 1-acc.*

(iii) *Every finitely generated free right R-module satisfies pan-acc.*

(iv) *For each positive integer n, every finitely generated free right $\mathcal{M}_n(R)$-module satisfies pan-acc.*

Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (iii): Let m be any positive integer and let $F = R_R^{(m)}$. Let n be any positive integer. By hypothesis, the ring $\mathcal{M}_{m+n}(R)$ satisfies right 1-*acc*. By Corollary 1.6, $R_R^{(m+n)}$ satisfies $(m+n)$-*acc*. Hence F satisfies n-acc. It follows that F satisfies pan-acc. This proves (iii).

(iii) $\Rightarrow$ (iv): By Lemma 1.8.

(iv) $\Rightarrow$ (i): Clear. □

Renault [11] gives an example of a right Noetherian ring R with the property that if F is the free right R-module of countably infinite rank then F does not satisfy 1-*acc*. Thus every finitely generated free right R-module is Noetherian and hence satisfies *pan-acc* but not every free right R-module satisfies *pan-acc*. If we assume in Theorem 1.9 that the ring R has additional properties then we can say more.

Corollary 1.10. *Let R be a right Goldie ring which satisfies dcc on right annihilators. Then the following statements are equivalent:*

(i) *For each positive integer n, the ring $\mathcal{M}_n(R)$ satisfies right pan-acc.*

(ii) *For each positive integer n, the ring $\mathcal{M}_n(R)$ satisfies right 1-acc.*

(iii) *Every free right R-module satisfies pan-acc.*

(iv) *For each positive integer n, every free right $\mathcal{M}_n(R)$-module satisfies pan-acc.*

Proof. By Theorem 1.9 and [4, Theorem 1].

In particular, Corollary 1.10 holds for any right nonsingular right Goldie ring (see [4] or [3, Theorem 1.5]).

Lemma 1.11. *Let T be a ring, let e be an idempotent in T and let R be the subring eTe of T. Let n be any positive integer.*

(i) *If T satisfies right n-acc then so too does R.*

(ii) *If every (finitely generated) free right T-module satisfies n-acc then so too does every (finitely generated) free right R-module.*

Proof. (i) See [5, Proposition 4.6].

(ii) Let F be any free right R-module. Without loss of generality we can take $F = R_R^{(\Lambda)}$, for some index set Λ. We can think of F as an R-submodule of the free right T-module $G = T_T^{(\Lambda)}$, in a natural way. Let N be any n-generated R-submodule of F, say $N = x_1R + \cdots + x_nR$. Then

$$NT = x_1RT + \cdots + x_nRT = x_1T + \cdots + x_nT \subseteq NT,$$

so that NT is an n-generated T-submodule of G.

Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be any ascending chain of n-generated R-submodules of F. Then, by the above remarks, $N_1T \subseteq N_2T \subseteq N_3T \subseteq \cdots$ is an ascending chain of n-generated T-submodules of G. By hypothesis, there exists a positive integer k such that $N_kT = N_{k+1}T = N_{k+2}T = \cdots$. Let $i \geq k$. Then $N_k = N_kR = N_k(eTe) = N_kTe = N_iTe = N_i$. That is, $N_k = N_{k+1} = N_{k+2} = \cdots$. It follows that F satisfies *n-acc*. □

Theorem 1.12. *Let R be a ring such that every (finitely generated) free right R-module satisfies pan-acc. Let T be a ring Morita equivalent to R. Then every (finitely generated) free right T-module satisfies pan-acc.*

Proof. By Lemmas 1.8 and 1.11. □

Let R be a ring which satisfies right *pan-acc* and let T be a ring Morita equivalent to R. Does T satisfy right *pan-acc*? By Theorem 1.12, this is certainly the case if every finitely generated free right R-module satisfies *pan-acc*. Heinzer and Lantz conjecture that if a ring R satisfies right *pan-acc* then every finitely generated free right R-module satisfies *pan-acc*, but this is still open according to Bonang [5] (see also [6, Ex. 0.1]). We shall return to this question in the next section.

2. Domains with *n-acc*

The purpose of this section is to give a proof of the main result of this paper, namely:

Theorem 2.1. *Let R be a left and right Ore domain and let n be a positive integer such that the free right R-module $R_R^{(n)}$ satisfies n-acc. Then every free right R-module satisfies n-acc.*

Combining this theorem with our remarks at the end of the previous section we see that if R is a left and right Ore domain such that for every positive integer n, the free right R-module $R_R^{(n)}$ satisfies *n-acc* then every ring Morita equivalent to R satisfies right *pan-acc*.

In order to prove Theorem 2.1 we first prove a number of lemmas.

Lemma 2.2. *Let D be a division ring and let $a \in \mathcal{M}_{m\times n}(D)$ where m and n are positive integers and $m > n$. Then there exists a unit p in $\mathcal{M}_m(D)$ such that the last $(m - n)$ rows of pa are all zero.*

Proof. The result is trivial if $a = 0$. Suppose that $a \neq 0$. Suppose that $n = 1$. Then

$$a = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

Now $a_{i1} \neq 0$ for some $1 \leq i \leq m$. If $i = 1$ let $p_1 = a_{11}^{-1} e_{11} + \sum_{k \neq 1} e_{kk}$; otherwise let

$$p_1 = e_{i1} + a_{i1}^{-1} e_{1i} + \sum\nolimits_{k \neq i,1} e_{kk} \in \mathcal{M}_m(D).$$

Then p_1 is a unit in $\mathcal{M}_m(D)$ and $p_1 a$ has first entry 1. Thus without loss of generality $a_{11} = 1$. Now let

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -a_{21} & 1 & 0 & 0 & \cdots & 0 \\ -a_{31} & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_{m1} & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_m(D).$$

Then p is a unit in $\mathcal{M}_m(D)$ with inverse

$$p^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & 0 & \cdots & 0 \\ a_{31} & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Moreover,

$$pa = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This proves the result when $n = 1$.

Now suppose that $n \geq 2$. Let a_1 be the $m \times (n-1)$ matrix over D and let b be the $m \times 1$ matrix over D such that $a = [a_1 | b]$ (in the obvious notation). By induction there exists a unit q_1 in $\mathcal{M}_m(D)$ such that $q_1 a_1$ has last $m - (n-1)$ rows zero. It follows that

$$q_1 a = [q_1 a_1 | q_1 b] = \begin{bmatrix} c & d \\ e & f \end{bmatrix},$$

where c, d, e, f are, respectively, an $(n-1) \times (n-1)$ matrix, an $(n-1)\times$ matrix, the zero $(m-(n-1)) \times (n-1)$ matrix and an $(m-(n-1)) \times 1$ matrix over D. If $f = 0$ then the result is proved. If $f \neq 0$ then we can argue as in the case $n = 1$ to produce a unit q_2 in $\mathcal{M}_m(D)$ such that

$$q_2 q_1 a = \begin{bmatrix} c & d \\ e & g \end{bmatrix}$$

where g is the $(m-(n-1)) \times 1$ matrix with first entry 1 and all other entries zero. Thus, if $p = q_2 q_1$ then p is a unit in $\mathcal{M}_m(D)$ and the last $(m-n)$ rows of pa are zero, as required. □

The proof of the next result is quite elementary. Recall that if R is a left Ore domain with left quotient division ring D then any element in D can be written in the form $c^{-1}r$ where $r \in R$, $0 \neq c \in R$. It is well known that if n is a positive integer and $q_i \in D$ $(1 \leq i \leq n)$ then there exist $r_i \in R$ $(1 \leq i \leq n)$, $0 \neq d \in R$ such that $q_i = d^{-1} r_i$ $(1 \leq i \leq n)$. This gives the following result.

Lemma 2.3. *Let R be a left Ore domain with left quotient division ring D and let m be a positive integer. Let p be any unit in $\mathcal{M}_m(D)$. Then there exists a nonzero element c in R such that $cp \in \mathcal{M}_m(R)$.*

In the next result we return to the situation considered in Section 1. Let R be any ring and let m and n be positive integers. Let S denote the ring $\mathcal{M}_n(R)$ and let α be the mapping from the collection of n-generated submodules of the free right R-module $F = R_R^{(m)}$ to the collection of cyclic S-submodules of $\mathcal{M}_{m\times n}(R)$, as defined in Section 1.

Lemma 2.4. *With the above notation, let $N \subseteq L$ be n-generated R-submodules of F such that N is an essential submodule of L. Then $\alpha(N)$ is an essential S-submodule of $\alpha(L)$.*

Proof. Let $L = (b_{11}, \dots, b_{m1})R + \cdots + (b_{1n}, \dots, b_{mn})R$, and let (b_{ij}) be the corresponding matrix in $\mathcal{M}_{m\times n}(R)$. Let $s = (c_{ij}) \in S$, where $c_{ij} \in R$ $(1 \leq i, j \leq n)$, such that $(b_{ij})s \neq 0$. There exists $1 \leq k \leq n$ such that the kth column of $(b_{ij})s$ is not zero. Thus

$$0 \neq x = (b_{11}, \dots, b_{m1})c_{1k} + \cdots + (b_{1n}, \dots, b_{mn})c_{nk} \in L.$$

There exists $r \in R$ such that $0 \neq xr \in N$. Let $t = re_{kk} \in S$. Then $0 \neq (b_{ij})st \in \alpha(N)$. It follows that $\alpha(N)$ is essential in $\alpha(L)$. □

We shall require the following special case of Lemma 2.4.

Corollary 2.5. *With the above notation, let R be a semiprime right Goldie ring. Let $N \subseteq L$ be n-generated R-submodules of F such that N is an essential submodule of L. Let a be any nonzero element of $\alpha(L)$. Then there exists a regular element c in R such that $ac^* \in \alpha(N)$, where c^* is the diagonal matrix in S with all diagonal entries c.*

Proof. Let $a \in (b_{ij})S$ (in the above notation). Let a_k $(1 \leq k \leq n)$ denote the columns of a. The proof of Lemma 2.4 shows that for each $1 \leq k \leq n$ there exists an essential right ideal E_k of R with

$$[0 \ \dots \ 0 \ a_k \ 0 \ \dots \ 0](E_k e_{kk}) \subseteq \alpha(N).$$

Let $E = E_1 \cap \cdots \cap E_n$. Then E is an essential right ideal of R and hence E contains a regular element c of R [9, 2.3.4 and 2.3.5]. Now

$$ac^* = [a_1 \ \dots \ a_n]c^* \in \alpha(N). \qquad \square$$

Proof of Theorem 2.1. Let R be a left and right Ore domain with quotient division ring D. Let n be a positive integer such that the free right R-module $R_R^{(n)}$ satisfies *n-acc*. To prove that every free right R-module satisfies *n-acc* it is sufficient to prove that every finitely generated free right R-module satisfies *n-acc* (see, for example, [3, Theorem 1.5]).

Let m be any positive integer. Let $F = R_R^{(m)}$. If $m \leq n$ then F satisfies *n-acc*. Suppose that $m \geq n+1$. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be any ascending chain of n-generated submodules of F. By [3, Lemma 1.1], for each $i \geq 1$, N_i has uniform dimension at most n. Thus, without loss of generality we can suppose that N_1 is essential in N_i for all $i \geq 1$.

By Corollary 1.2, $\alpha(N_1) \subseteq \alpha(N_2) \subseteq \alpha(N_3) \subseteq \cdots$ is an ascending chain of cyclic S-submodules of $\mathcal{M}_{m\times n}(R)$, where $S = \mathcal{M}_n(R)$. For each $i \geq 1$, let $a_i \in \mathcal{M}_{m\times n}(R)$ such that $\alpha(N_i) = a_iS$. By Lemmas 2.2 and 2.3 there exist a unit p in $\mathcal{M}_m(D)$ and a nonzero element c in R such that $cp \in \mathcal{M}_m(R)$ and cpa_1 has its last $(m-n)$ rows all zero. By Corollary 2.5, for each $i \geq 1$, there exists a nonzero element d_i in R such that $cpa_id_i^* \in cpa_1S$. Thus the last $(m-n)$ rows of $cpa_id_i^*$ are zero and hence the last $(m-n)$ rows of cpa_i are zero.

Consider the ascending chain $cpa_1S \subseteq cpa_2S \subseteq cpa_3S \subseteq \cdots$ in $\mathcal{M}_{m\times n}(R)$. By Corollary 1.3 $\beta(cpa_1S) \subseteq \beta(cpa_2S) \subseteq \beta(cpa_3S) \subseteq \cdots$ is an ascending chain of n-generated submodules of F. Moreover, for each $i \geq 1$, $\beta(cpa_iS)$ is contained in the submodule G of F consisting of all elements of F of the form $(r_1,\dots,r_n,0,\dots,0)$, where $r_i \in R$ $(1 \leq i \leq n)$. But $G \cong R_R^{(n)}$ and hence, by hypothesis, G satisfies *n-acc*. Thus there exists a positive integer k such that $\beta(cpa_kS) = \beta(cpa_{k+1}S) = \beta(cpa_{k+2}S) = \cdots$ By Lemma 1.4, if we now apply α we have $cpa_kS = cpa_{k+1}S = cpa_{k+2}S = \cdots$ Now using the fact that $c \neq 0$ and p is a unit, we have $a_kS = a_{k+1}S = a_{k+2}S = \cdots$ Finally applying β we obtain $N_k = N_{k+1} = N_{k+2} = \cdots$ It follows that F satisfies *n-acc*. $\square$

If in Theorem 2.1 the ring R is commutative we can do rather better, as the next result shows. If $a \in \mathcal{M}_n(R)$, for any commutative ring R and positive integer n, then $\det(a)$ will denote the determinant of a.

Theorem 2.6. *Let R be a commutative domain and let n be a positive integer such that the free R-module $R_R^{(n-1)}$ satisfies n-acc. Then every free R-module satisfies n-acc.*

Proof. In view of Theorem 2.1 it is sufficient to prove that the free R-module $F = R_R^{(n)}$ satisfies *n-acc*. Let $S = \mathscr{M}_n(R)$ and let D denote the field of fractions of R. By Corollary 1.6, it is sufficient to prove that S satisfies right 1-*acc*. Let $a_1S \subsetneq a_2S \subsetneq a_3S \subsetneq \cdots$ be any ascending chain of nonzero principal right ideals of S. By the proof of Theorem 2.1, we can suppose without loss of generality that a_1 has rank n, for otherwise there exists a unit p in $\mathscr{M}_n(D)$ and a nonzero element c in R such that cpa_i has zero last row for all $i \geq 1$. Now $\det(a_1)R \subseteq \det(a_2)R \subseteq \det(a_3)R \subseteq \cdots$, so there exists a positive integer k such that $\det(a_k)R = \det(a_{k+1})R = \det(a_{k+2})R = \cdots$.

Note that for all $i \geq k$, $a_k = a_ib_i$ for some $b_i \in \mathscr{M}_n(R)$ and $\det(a_k)R = \det(a_i)R$. Since $\det(a_k) = \det(a_i)\det(b_i) \neq 0$, it follows that $\det(b_i)$ is a unit in R and hence b_i is a unit in $S = \mathscr{M}_n(R)$. Thus $a_kS = a_{k+1}S = a_{k+2}S = \cdots$. It follows that F satisfies *n-acc*, as required. □

Nicolas [10, Proposition 1.4] proved that if R is a commutative domain which satisfies 1-*acc* then every free R-module satisfies 1-*acc*. Now Theorem 2.6 gives at once:

Corollary 2.7. *Let R be a commutative domain which satisfies 2-acc. Then every free R-module satisfies 2-acc.*

3. Rings whose free modules have *pan-acc*

In this section, our concern is to give, in the spirit of [2, 5], a range of examples of rings whose (finitely generated) free modules satisfy *n-acc*, for some positive integer n, or *pan-acc*. As noted earlier, Heinzer and Lantz [7] give examples, for each positive integer n, of a commutative ring R_n which satisfies *n-acc* but not $(n+1)$-*acc*, and hence not *pan-acc*.

Proposition 3.1. *Let R be a subring of a ring S and let A be an ideal of R such that A is a left ideal of S and the ring R/A is right perfect. Suppose further that there exists a positive integer n such that every (finitely generated) free right S-module satisfies n-acc. Then every (finitely generated) free right R-module satisfies n-acc.*

Proof. Let n be any positive integer. Let I be any nonempty index set and let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be any ascending chain of n-generated submodules of the free right R-module $R_R^{(I)}$. In a natural way we can think of $R_R^{(I)}$ as an R-submodule of the right S-module $F = S_S^{(I)}$.

Clearly $N_1S \subseteq N_2S \subseteq N_3S \subseteq \cdots$ is an ascending chain of n-generated S-submodules of F. By hypothesis, there exists a positive integer t such that $N_tS = N_{t+1}S = N_{t+2}S = \cdots$. Because A is a left ideal of S it follows that $N_tA = N_{t+1}A = N_{t+2}A = \cdots$.

Let $N = \bigcup_{i \geq 1} N_i$. Then $NA = N_tA$ and hence N/N_t is a right (R/A)-module. By the Jonah–Renault Theorem (see [8, Main Theorem; 11, Proposition 1.2]), N/N_t satisfies *n-acc* and hence there exists $s \geq t$ with $N_s = N_{s+1} = N_{s+2} = \cdots$. □

Now suppose that in Proposition 3.1, A is a finitely generated right ideal, rather than a left ideal, of S and that A_S is generated by k elements. In this case, in the proof of Proposition 3.1, $N_1A \subseteq N_2A \subseteq N_3A \subseteq \cdots$ is an ascending chain of (nk)-generated S-submodules of F. If F satisfies (nk)-*acc* then there exists a positive integer t such that $N_tA = N_{t+1}A = N_{t+2}A = \cdots$. By the proof of Proposition 3.1, it follows that $R_R^{(I)}$ satisfies n-*acc*. We have thus proved the following companion to Proposition 3.1.

Proposition 3.2. *Let R be a subring of a ring S and let A be an ideal of R such that A is a finitely generated right ideal of S and the ring R/A is right perfect. Suppose further that every (finitely generated) free right S-module satisfies pan-acc. Then every (finitely generated) free right R-module satisfies pan-acc.*

Proposition 3.3. *Let T be a subring of a ring S and let B and C be ideals of T such that the rings T/B and T/C are right perfect and C is a finitely generated right ideal of T. Let R_1 and R_2 denote the subrings $T + SB$ and $T + CS$ of S, respectively.*

(i) *If n is a positive integer such that every (finitely generated) free right S-module satisfies n-acc then so too does every (finitely generated) free right R_1-module.*

(ii) *If every (finitely generated) free right S-module satisfies pan-acc then so too does every (finitely generated) free right R_2-module.*

Proof. (i) Note that SB is a left ideal of S and a two-sided ideal of R_1 such that $R_1/SB \cong T/(T \cap SB)$ which is right perfect, being a homomorphic image of T/B. Apply Proposition 3.1 to obtain that every free right R_1-module satisfies n-*acc*.

(ii) Similar to (i). □

Proposition 3.4. *Let S be any ring and let n be a positive integer such that every (finitely generated) free right S-module satisfies n-acc. Let T be a subring of S and let B be an ideal of T such that the ring T/B is right perfect. Let L be any left ideal of S such that $B + LB$ is a left ideal of S and let $R = T + LB$. Then every (finitely generated) free right R-module satisfies n-acc.*

Proof. Let $A = B + LB$. Then A is a left ideal of S and a two-sided ideal of R such that $R/A = (T + LB)/(B + LB) \cong T/(B + (T \cap LB))$ which is right perfect, being a homomorphic image of T/B. Apply Proposition 3.1. □

Corollary 3.5. *Let S be any ring and let n be a positive integer such that every (finitely generated) free right S-module satisfies n-acc. Let T be a subring of S and let B be an ideal of T such that the ring T/B is right perfect. Let L be any left ideal of S such that $S = T + L$ and let $R = T + LB$. Then every (finitely generated) free right R-module satisfies n-acc.*

Proof. Because $S = T + L$, $B + LB = SB$ is a left ideal of S. Apply Proposition 3.4. □

The next result is a companion to Corollary 3.5.

Proposition 3.6. *Let S be any ring such that every (finitely generated) free right S-module satisfies pan-acc. Let T be a subring of S and let B be an ideal of T such that B is finitely generated as a right ideal and the ring T/B is right perfect. Let E be any right ideal of S such that $S = T + E$ and let $R = T + BE$. Then every (finitely generated) free right R-module satisfies pan-acc.*

Proof. There exist a positive integer m and elements $b_i \in B$ such that $B = b_1T + \cdots + b_mT$. Now $B + BE = BS = b_1S + \cdots + b_mS$. Now apply Proposition 3.2. □

Let S be a ring and let A be a right ideal of S. Then we define $\mathscr{I}(A) = \{s \in S\colon sA \subseteq A\}$. Then $\mathscr{I}(A)$ is the biggest subring of S in which A is a two-sided ideal and $\mathscr{I}(A)$ is called the *idealizer of A in S*. If A is a left ideal we can construct the idealizer $\mathscr{I}(A)$ in a similar way.

Proposition 3.7. *Let A be a left or right ideal of a ring S, let T be a right perfect subring of $\mathscr{I}(A)$ and let $R = T + A$.*

(i) *If A is a left ideal and n is a positive integer such that every (finitely generated) free right S-module satisfies n-acc then so too does every (finitely generated) free right R-module.*

(ii) *If A is a finitely generated right ideal and every (finitely generated) free right S-module satisfies pan-acc then so too does every (finitely generated) free right R-module.*

Proof. (i) By Proposition 3.1 since $R/A \cong T/(T \cap A)$ which is right perfect.

(ii) Similar to (i). □

Corollary 3.8. *Let T be a right perfect subring of a ring S, let A be an ideal of S and let $R = T + A$. If n is a positive integer such that every (finitely generated) free right S-module satisfies n-acc then so too does every (finitely generated) free right R-module.*

Proof. By Proposition 3.7, for in this case $S = \mathscr{I}(A)$. □

We next mention an interesting special case of Proposition 3.7.

Proposition 3.9. *Let A be a left or right ideal of a ring S such that the S-module S/A has finite composition length. Let $R = \mathscr{I}(A)$.*

(i) *If A is a left ideal and n is a positive integer such that every (finitely generated) free right S-module satisfies n-acc then so too does every (finitely generated) free right R-module.*

(ii) *If A is a finitely generated right ideal and every (finitely generated) free right S-module satisfies pan-acc then so too does every (finitely generated) free right R-module.*

Proof. The ring R/A is isomorphic to the endomorphism ring of the S-module S/A and hence R/A is semiprimary, whence right perfect, by [1, 28.8 and 29.3]. Apply Propositions 3.1 and 3.2. □

Now we introduce some matrix examples. First, we prove the following result.

Proposition 3.10. *Let A and B be ideals of a ring R such that $AB = 0$, the ring R/B is right perfect and every (finitely generated) free right (R/A)-module satisfies n-acc, for some fixed positive integer n. Then every (finitely generated) free right R-module satisfies n-acc.*

Proof. Let F be a (finitely generated) free right R-module and let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be any ascending chain of n-generated submodules of F. Then $(N_1 + FA)/FA \subseteq (N_2 + FA)/FA \subseteq (N_3 + FA)/FA \subseteq \cdots$ is an ascending chain of n-generated submodules of the (finitely generated) free right (R/A)-module F/FA. By hypothesis, there exists a positive integer k such that $N_k + FA = N_{k+1} + FA = N_{k+2} + FA = \cdots$. Now $AB = 0$ gives $N_kB = N_{k+1}B = N_{k+2}B = \cdots$. The argument of Proposition 3.1 now gives that $N_t = N_{t+1} = N_{t+2} = \cdots$ for some integer $t \geq k$. Thus F satisfies *n-acc*. □

Let S and T be rings and let M be a left S-, right T-bimodule. Let $[S, M; 0, T]$ denote the set of "matrices"

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix},$$

where $s \in S$, $m \in M$ and $t \in T$. Denote the above matrix by $[s, m; 0, t]$. Then $[S, M; 0, T]$ is a ring with respect to the usual definitions of matrix addition and multiplication.

Corollary 3.11. *Let S be a ring such that every (finitely generated) free right S-module satisfies n-acc, for some fixed positive integer n. Let T be a right perfect ring and let M be a left S-, right T-bimodule. Let $R = [S, M; 0, T]$. Then every (finitely generated) free right R-module satisfies n-acc.*

Proof. Let $A = [0, M; 0, T]$ and $B = [S, M; 0, 0]$. Then A and B are ideals of R, $AB = 0$, $R/A \cong S$ and $R/B \cong T$. Apply Proposition 3.10. □

Corollary 3.11 has the following immediate consequence.

Corollary 3.12. *Let K be a field and let S be a K-algebra such that every (finitely generated) free right S-module satisfies n-acc, for some fixed positive integer n. Let M be any left S-module and let $R = [S, M; 0, K]$. Then every (finitely generated) free right R-module satisfies n-acc.*

Corollary 3.13. *Let K be a field and let S be a right and left Noetherian K-algebra. Let M be any left S-module and let $R = [S, M; 0, K]$. Then every free right R-module satisfies pan-acc.*

Proof. By Corollary 3.12 and [11, Corollaire 3.3]. □

Example 3.14. Let K be a field and let S be a simple right and left Noetherian K-algebra which is not Artinian, let U be any simple left S-module and let $R = [S, U; 0, K]$. Then

(i) R is a left Noetherian ring, every finitely generated free left R-module is Noetherian but not every free left R-module satisfies 1-*acc*.

(ii) Every free right R-module satisfies *pan-acc*.

Proof. (i) By [9, 1.1.7] and [11, Proposition 3.4].

(ii) By Corollary 3.13. □

Now taking K, S as in Example 3.14 and U a simple right S-module, let R denote the ring $[K, U; 0, S]$. Let $A = [0, U; 0, S]$ and $B = [K, U; 0, 0]$. Then A and B are ideals of R and $AB = 0$. Moreover. $R/A \cong K$, so that R/A is right perfect, $R/B \cong S$, so that every free right (R/B)-module satisfies *pan-acc* [11, Corollaire 3.3], but not every free right R-module satisfies 1-*acc* [11, Proposition 3.4]. Compare Proposition 3.10. Note also that in this case if $C = BA = [0, U; 0, 0] \neq 0$, then $C^2 = 0$ and $R/C \cong S \oplus K$, so that every free right (or left) (R/C)-module satisfies *pan-acc*.

Many more examples can be produced using Corollary 3.11. For example, let S be a commutative Noetherian domain with field of fractions L, let K be any extension field of L and let V be any vector space over K. Then the ring $R = [S, V; 0, K]$ has the property that every free right R-module satisfies *pan-acc* (Corollary 3.11 and [11, Corollaire 2.3]). Note that R is right Noetherian if and only if R is right Goldie if and only if V is finite dimensional over K [9, 1.1.7].

Our next aim is to give an example of a commutative domain R such that every free R-module satisfies *pan-acc* but the polynomial ring $R[t]$ does not satisfy 2-*acc*. In contrast we have the following elementary fact.

Proposition 3.15. *Let R be a domain which satisfies right* 1-*acc. Then the polynomial ring $R[t]$ satisfies right* 1-*acc.*

Proof. Let S denote the ring $R[t]$. For any polynomial $f(t)$ in S, let $\delta(f(t))$ denote the degree of $f(t)$ and, if $f(t) \neq 0$, let $\lambda(f(t))$ denote the leading coefficient of $f(t)$.

Let $f_1(t)S \subseteq f_2(t)S \subseteq f_3(t)S \subseteq \cdots$ be any ascending chain of principal right ideals of S. Then $\delta(f_1(t)) \geq \delta(f_2(t)) \geq \delta(f_3(t)) \geq \cdots$, so that without loss of generality we can suppose that all the polynomials $f_i(t)$ are nonzero with the same degree.

Moreover, $\lambda(f_1(t))R \subseteq \lambda(f_2(t))R \subseteq \lambda(f_3(t))R \subseteq \cdots$ so there exists a positive integer n with $\lambda(f_n(t))R = \lambda(f_{n+1}(t))R = \lambda(f_{n+2}(t))R = \cdots$. It is now easy to check that $f_n(t)S = f_{n+1}(t)S = f_{n+2}(t)S = \cdots$. Thus S satisfies right 1-*acc*. □

Example 3.16. Let K/L be a nonalgebraic field extension and let R denote the subring $L + xK[x]$ of the polynomial ring $K[x]$. Then R is a commutative domain such that every free R-module satisfies *pan-acc* but the polynomial ring $R[t]$ does not satisfy 2-*acc*.

Proof. Let T denote the ring $R[t]$. Note first that every free R-module satisfies *pan-acc* by Corollary 3.8 (take $S = K[x]$, $T = L$ and $A = xK[x]$). There exists an element a in K such that a is not algebraic over L. For each positive integer n,

$$x^2a^n = (x^2a^{n+1})t - (xat - x)xa^n \in (x^2a^{n+1},\ xat - x).$$

Consider the chain of 2-generated ideals of T:

$$(x^2a, xat - x) \subseteq (x^2a^2, xat - x) \subseteq (x^2a^3, xat - x) \subseteq \cdots. \tag{1}$$

Now suppose that $x^2a^{n+1} \in (x^2a^n,\ xat - x)$, for some positive integer n. There exist u, v in T such that

$$x^2a^{n+1} = x^2a^n u + (xat - x)v.$$

Setting $t = 1/a$, we have

$$x^2a^{n+1} = x^2a^n(d_0 + d_1(1/a) + d_2(1/a)^2 + \cdots + d_m(1/a)^m)$$

for some $m \geq 1$, $d_i \in R$ $(0 \leq i \leq m)$. Hence

$$a^{m+1} = d_0a^m + d_1a^{m-1} + \cdots + d_m.$$

For each $0 \leq i \leq m$, there exist $c_i \in L$, $f_i(x) \in K[x]$ such that $d_i = c_i + xf_i(x)$. It follows that $a^{m+1} = c_0a^m + c_1a^{m-1} + \cdots + c_m$, a contradiction. Thus every inclusion in the chain (1) is proper and hence the ring T does not satisfy 2-*acc*. □

Note that in Example 3.16 the ring $R[t]$ is isomorphic to the subring $L[t] + xK[x,t]$ of the polynomial ring $S = K[x,t]$. The ring S is a commutative Noetherian domain and every free S-module satisfies *pan-acc*. [11, Corollaire 2.3]. Moreover, the ring $R[t]$ has as a subring the ring $S' = L + xK[x,t]$. By Corollary 3.8 every free S'-module satisfies *pan-acc*. This indicates how vital it is to have a right perfect subring involved in the constructions in this section.

4. Torsionless modules

Let R be a ring and let M be a right R-module. The module M is called *torsionless* provided for each $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M,R)$ such that $f(m) \neq 0$ (see, for example, [9, 3.4.2]). It is easy to see that this is equivalent to saying that M embeds in a direct product of copies of R_R. Note that if U is any left R-module then the right R-module $\mathrm{Hom}_R(U,R)$ is torsionless (see [9, 3.4.2]).

Let R be a right Noetherian right nonsingular ring. Then every torsionless right R-module satisfies *pan-acc* (see [3, Theorem 1.5] or [11, Corollaire 2.3]). In this section we shall give some examples of rings which need not be right Noetherian but for which every torsionless right module satisfies *pan-acc*.

Proposition 4.1. *Let R be a subring of a ring S and let A be an ideal of R such that A is a left ideal of S and the ring R/A is right perfect. Suppose further that there exists a positive integer n such that every torsionless right S-module satisfies n-acc. Then every torsionless right R-module satisfies n-acc.*

Proof. By the proof of Proposition 3.1 with the direct product $(R_R)^I$ replacing the direct sum $(R_R)^{(I)}$. □

In a similar way, the proof of Proposition 3.2 can be adapted to give:

Proposition 4.2. *Let R be a subring of a ring S and let A be an ideal of R such that A is a finitely generated right ideal of S and the ring R/A is right perfect. Suppose further that every torsionless right S-module satisfies pan-acc. Then every torsionless right R-module satisfies pan-acc.*

Corollary 4.3. *Let R be a subring of a right Noetherian right nonsingular ring S and let A be an ideal of R such that A is a left or right ideal of S and the ring R/A is right perfect. Then every torsionless right R-module satisfies pan-acc.*

Proof. By Propositions 4.1 and 4.2 and [11, Corollaire 2.3]. □

Another consequence of Propositions 4.1 and 4.2 is the following result.

Corollary 4.4. *Let T be a right Noetherian right nonsingular ring and let B be any ideal of T such that the ring T/B is right Artinian. Let R denote the subring $T+xB[x]$ of the polynomial ring $T[x]$. Then every torsionless right R-module satisfies pan-acc.*

Proof. If E is an essential right ideal of the polynomial ring $S = T[x]$ then the set E' of leading coefficients of the elements of E, together with 0, forms an essential right ideal of T. It follows that the ring S is right Noetherian right nonsingular. Let L denote the ideal xS of S. Note that $S = T + L$. Let $A = B + xB[x] = SB$. Then $A \subseteq R$ and A is an ideal of S. Moreover, the ring R/A is a homomorphic image of T/B, so is right Artinian. By Corollary 4.4, every torsionless right R-module satisfies *pan-acc*. □

It is now clear that the results of Section 3 can be adapted to give corresponding results for torsionless modules. We now prove an analogue of Proposition 3.10.

Proposition 4.5. *Let A and B be ideals of a ring R such that $AB = 0$, the ring R/B is right perfect and every torsionless right (R/A)-module satisfies n-acc, for some fixed positive integer n. Then every torsionless right R-module satisfies n-acc.*

Proof. Let $F = (R_R)^I$, for any nonempty index set I. Let $R' = R/A$ and $A^* = A^I$. Then $F/A^* \cong (R'_{R'})^I$, which is a torsionless right R'-module. Now the result follows by the proof of Proposition 3.10 because $A^*B = 0$. □

Corollary 4.6. *Let K be a field and let S be a right Noetherian right nonsingular K-algebra. Let M be any left S-module and let $R = [S, M; 0, K]$. Then every torsionless right R-module satisfies pan-acc.*

Proof. This result follows from Proposition 4.5 in essentially the same way that Corollary 3.12 follows from Proposition 3.10, by using [11, Corollaire 2.3]. □

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